Pricing of Presale Contracts with Macroeconomic Factors and Stochastic Basis Risk

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Abstract

The extent to which the covariance structure of housing prices, macroeconomic factors and stochastic basis price affect presale price has not been studied. This paper extends the work of Chang and Ward (1993) and Leung, et al. (2007) by investigating the relationship between presale price and housing prices using unbiased futures pricing and taking into account stochastic basis risk. Using linear regressions to model the relationship between housing prices and macroeconomic factors and following Wang and Wu (2011) in assuming that the stochastic basis price functions as a modified Brownian bridge process, we provide the closed-form solutions for presale price with basis risk. Furthermore, we examine the impact of housing prices, interest rates and macroeconomic factors on the basis price and presale price. Ignoring the effects of macroeconomic factors (e.g., land price, construction costs, inflation rates and the stock price index) on basis price leads to underestimate presale pricing, especially during periods characterized by negative initial basis.

Keywords: Presale price, macroeconomic factors, basis price, modified Brownian bridge process
1. Introduction

Presale pricing is a common practice in real estate markets; it is the sale of housing development units before completion (and in some cases, even before construction has begun). Some studies (e.g., Chang and Ward (1993); Hua, Chang and Hsieh (2001); Wong, Yiu, Tse and Chau (2006)) treat a presale as a forward or futures contract. In futures pricing for presale contracts, changes in housing prices and stochastic basis risk are crucial factors. In addition, numerous studies have identified a significant relationship between housing prices and macroeconomic factors (see Leung (2004) for a review of macroeconomics and housing). The influence of the covariance structure of housing prices and macroeconomic factors and that of stochastic basis risk on presale price has not been extensively studied. Therefore, the main objective of this paper is to investigate the relationship between presale price and housing prices using unbiased futures pricing with stochastic basis risk. Assuming that the stochastic basis price functions as a modified Brownian bridge process, we provide closed-form solutions for presale price with basis risk and further examine the impacts of housing prices, interest rates, and macroeconomic factors on basis price and presale price.

Chan et al. (2008) suggest that although research on Asian property markets has increased, the presale method has not received significant attention in the academic literature. Despite its complexity, the presale method has been popular in many Asian countries for at least five decades. Recently, the presale method gained attention in the U.S. as residential property prices surged during 2005-2006. Some early research treats a presale as a forward or futures contract. For instance, Chang and Ward (1993); Hua, Chang and Hsieh (2001); and Wong, Yiu, Tse and Chau (2006) use this approach and analyze the relationship between presale price and future spot sale price in investigating the impact of the presale market on housing supply.
adjustment. Chang and Ward (1993) further discuss the “basis price” (defined as the presale price minus spot price) and analyze how it varies based on expectations regarding the housing cycle, the quality/location of the housing and other product characteristics. Leung et al. (2007) use a linear regression to model the presale contract price in terms of a few independent variables, such as the market price of existing condominiums, the interest rate and hidden forward risks.

The second strand of the literature treats a presale as a call option contract and addresses issues such as the impact of the presale on a developer’s (or purchaser’s) development strategies and on pricing. Lai et al. (2004) lay the foundation for a real option and examined the problem from the perspective of the purchaser and the developer of a condominium unit in an environment in which both parties demanded a risk-free rate of return on their investments. They analyze property price data from Shanghai, Hong Kong and Taipei and surmise that property prices could be modeled using the Geometric Brownian Motion (GBM) method. Chan et al. (2008) also use the options approach and extend the literature by applying a simple equilibrium model to presale contracts under the assumption that property prices are normally distributed. Choi et al. (2010) assume that property prices follow GBM rather than a normal distribution. Choi et al. (2010) also allow for uncertainty with regard to construction cost by treating it as a stochastic variable, whereas Chan et al. assume that construction costs are deterministic. Almost all of the existing models for pricing presale contracts assume that the property price process follows GBM. However, this assumption only considers the randomness of the property prices themselves; it ignores co-movement between property prices and macroeconomic factors. Many studies empirically highlight the significant

relationship between macroeconomic factors and real estate returns; see Leung (2004) for a review of macroeconomics and housing. For instance, McCue and Kling (1994) show that macroeconomic factors explain nearly 60% of the series variation for real estate. Green (2002) and Chen and Patel (2002) also demonstrate that stock return dynamics are a key determinant of housing price trends. In addition, from a housing supply perspective, construction cost and land acquisition significantly influence the cost of houses. Jud and Winkler (2002) find that appreciation in real housing prices is strongly positively related to increases in construction costs. Grimes and Aitken (2010) indicate that an increase in land cost or construction costs can reduce the new housing supply, which in turn can increase housing prices.

This paper contributes to the current literature on pricing for presale contracts. First, we extend the work of Chang and Ward (1993) and Leung et al. (2007) in modeling the co-movement of housing prices and macroeconomic factors. In so doing, we follow Wang and Wu (2011) by assuming that the stochastic basis price follows a modified Brownian bridge process, and we provide closed-form solutions for basis price and presale price using linear regression. Notably, the empirical results presented by Chang et al. (2012) indicate that the statistically significant macroeconomic factors that determine housing price returns include land price returns, construction costs, the inflation rate and the stock price index. The sensitivity findings show that the basis price is an increasing function of the correlation coefficient between housing prices (macroeconomic factors) and the basis price; however, the sign of this correlation coefficient determines the relationship between the basis volatility and the basis price. Furthermore, we find that the effect of time to maturity on the basis price mainly depends on the ratio of the initial basis to time to maturity. On the other hand, the presale price is a decreasing function of the correlation coefficient for housing prices (macroeconomic factors) and the basis price; however, the sign of the correlation coefficient
between housing prices (macroeconomic factors) and the basis price determines the relationship between the basis volatility and the presale price. Finally, ignoring the impacts of macroeconomic factors (e.g., land price, construction costs, inflation rates and the stock price index) on basis price leads to the underestimation of presale price, especially during periods characterized by negative initial basis.

This paper is organized as follows. Section 2 illustrates the housing price model and presents the macroeconomic dynamics and the interest rate. Section 3 presents the closed-form formulas for basis risk and the presale price. Section 4 discusses the sensitivity analysis. Section 5 summarizes the paper and presents the conclusions drawn.
2. Model Setup

To examine the basis process and presale price, this section first provides a multiple linear regression model that describes the effect of macroeconomic factors on housing prices and then presents a stochastic interest rate process that can be used to consider the relationship between housing prices and interest rates.

Some empirical studies (e.g., Abraham and Hendershott (1996), Muellbauer and Murphy (1997), Leung (2004) and Oikarinen (2009)) have indicated that a significant relationship exists between housing prices and macroeconomic factors. Below, we follow Chang et al. (2012) in employing a linear regression to capture the co-movement of macroeconomic factors and housing prices and assuming that the macroeconomic factors at play follow geometric Brownian motion. The housing price return $H$ on a filtered probability space $(\Omega, F, P, (F_t)_{t=0}^T)$, is governed by the following process:

$$\frac{dH(t)}{H(t)} = \left( \mu_H + \sum_{i=1}^{n} c_i \frac{dM_i(t)}{M_i(t)} \right) dt + \sigma_H dW_H(t),$$

where $P$ is the physical probability measure (i.e., the real-world probability measure); $F_t, t \in [0, T]$ is the smallest sigma field such that $H(t)$ and $M_i(t)$ are known and measurable; and $(F_t)_{t=0}^T$ is the right-continuous natural filtration, that is, $F_t \subset F_s, t \leq s$. Each macroeconomic factor $M_i, i = 1, ..., n$ follow the geometric Brownian motion as follows:

$$\frac{dM_i(t)}{M_i(t)} = \mu_{M_i} dt + \sigma_{M_i} dW_{M_i}(t),$$

where $\mu_H$ and $\mu_{M_i}, i = 1, 2, ..., n$ are drift terms for the housing price and macroeconomic factors, respectively. $c_i$ denotes the sensitivity parameter for the macroeconomic factor $M_i$. 


and the housing price. The variances $\sigma^2_H$ ($\sigma^2_{M_i}$) denote the residual variance for housing prices ($i^{th}$ macroeconomic factor), and $W_H(t)$ and $W_{M_i}(t)$ is a Wiener process for the housing price and macroeconomic factors, respectively, such that

$$E\left(W_H(t)W_{M_i}(t)\right) = 0, \quad E\left(W_{M_i}(t)W_{M_j}(t)\right) = \rho_{M_i,M_j}(t), \quad i, j = 1, 2, ..., n.$$  (3)

The dynamics process for the interest rate can be written as

$$dr(t) = \left[\tilde{k} - \tilde{\theta} - \frac{\sigma^2_H}{2} - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \rho_{M_i,M_j} \sigma_{M_i} \sigma_{M_j}\right] dt + \tilde{\sigma}_H dW_Q(t) + \sum_{i=1}^{n} \tilde{\sigma}_{M_i} dW_{Q,i}(t),$$  (4)

where $\tilde{k}$ denotes the mean-reverting force measurement, $\tilde{\theta}$ represents the long-run mean of the interest rate, $\tilde{\sigma}_r$ is the volatility parameter for the interest rate, and $W_r(t)$ is a Wiener process for the interest rate under the physical probability measure $P$. For the sake of simplicity, we assume that $E\left(W_H(t)W_r(t)\right) = \rho_{Hr}$, $E\left(W_{M_i}(t)W_r(t)\right) = 0$.

Using methods developed by Rubinstein (1976) and Brennan (1979), the discount values for both of the housing prices become martingales under a risk-neutral valuation relationship. Given the risk-neutral measure $Q$, the expected returns for housing prices shift according to the risk-free interest rate, but the covariance structure remains unchanged. Through algebraic rearrangement, housing prices under the risk-neutral measure $Q$ can be obtained from the following expressions:

$$\frac{dH(t)}{H(t)} = \left\{r(t) - \frac{1}{2} \sigma^2_H - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \rho_{M_i,M_j} \sigma_{M_i} \sigma_{M_j}\right\} dt + \tilde{\sigma}_H dW_Q(t) + \sum_{i=1}^{n} \tilde{\sigma}_{M_i} dW_{Q,i}(t),$$  (5)

and the dynamics of the interest rate under the risk-neutral measure $Q$ are as follows:

$$d\tilde{k} = \left[\theta - \tilde{k}\right] dt + \tilde{\sigma}_{k} dW_Q$$  (6)
\[
\tilde{\sigma}_H = \sigma_H \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{\sigma}_r = \sigma_r \times \begin{bmatrix} \rho_{Hr} \\ \sqrt{1 - \rho_{Hr}^2} \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{\sigma}_M_i = \sigma_{M_i} \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},
\]

where \( k \) and \( \theta \) denote the mean-reverting force and the long-run mean of the interest rate under the risk-neutral measure \( Q \), respectively. The correlation coefficient \( \rho_{Hr} \) shows the relationship between the housing price and the interest rate, and \( W^Q(t) \) stands for a four-dimensional standard Brownian motion with a filtered probability space \( (\Omega, F, Q, (F_t)_{t \geq 0}) \).

According to Equation (5), the housing price return volatility is

\[
\text{Var}\left( \frac{dH(t)}{H(t)} \right) \equiv V(t) = \tilde{\sigma}_H^2 + C \Sigma(t) C',
\]

where \( C = [c_1, \ldots, c_n] \), \( \sigma_{M_{ij}}(t) = \rho_{M_{ij}}(t) \sigma_{M_i}(t) \sigma_{M_j}(t) \), \( \Sigma(t) = \left[ \sigma_{M_{ij}}(t) \right]_{\text{non}}, \quad i, j = 1, 2, \ldots, n \).

Hence, the total volatility of the housing price \( V(t) \) includes two components: the covariance matrix of the macroeconomic factors \( \alpha(t) \) and the residual housing price volatility \( \tilde{\sigma}_H^2 \). In addition to the volatilities \( \tilde{\sigma}_H \) and \( \sigma_{M_i} \), the housing price return is related to the coefficient \( c_j \), which gauges the impact of macroeconomic factor \( M_j \) on the housing price and the
covariance structure of the macroeconomic factors.

Here, \( f_p(t,T) \) is the forward price at time \( t \), with maturity \( T \) satisfying

\[
f_p(t,T) = \frac{H(t)}{B(t,T)}, \quad t \leq T,
\]

(7)

where \( B(t,T) \) is the time \( t \) price of a zero coupon bond that pays one dollar at time \( T \).

Assume that a unique forward risk-neutral measure \( P_T \) exists that is given by the following Radon-Nikodym derivative:

\[
\frac{dP_T}{dQ} = \exp \left\{ \int_0^T b(u,T) \cdot dW^T(u) - \frac{1}{2} \int_0^T |b(u,T)|^2 \, du \right\},
\]

(8)

\[
b(u,T) = \tilde{\sigma}_r / \tilde{k} \left[ \exp(-\tilde{k}(T-u)) - 1 \right]
\]

(9)

where \(| \cdot |\) denotes the Euclidean norm in \( R^4 \). Using the Girsanov theorem, we obtain

\[
dW^T(t) = dW^Q(t) - b(t,T) \, dt, \quad \forall t \in [0,T],
\]

where \( W^T(t) \) represents the three-dimensional Brownian motion under the forward measure \( P_T \). Hence, under the forward measure \( P_T \), the forward price satisfies

\[
f_p(T,T) = f_p(0,0) \exp \left\{ -\frac{1}{2} \int_0^T \phi(u,T)^2 \, du + \int_0^T \phi(u,T) \cdot dW^T(u) \right\},
\]

(10)

where \( \phi(u,T) = \tilde{\sigma}_H + \Omega \sum^* C - b(u,T) \), \( \Omega = \left[ \rho_{M,M} \right]_{\text{diag}} \) is the variance-covariance matrix; \( \sum^* \) is a diagonal matrix that satisfies \( \sum^* = \text{diag}(\sigma_{M_1}, \ldots, \sigma_{M_n}) \).
3. The Basis and Presale Pricing Models

Working from a linear regression model that demonstrates the relationship between the housing price and macroeconomic factors, we use the modified Brownian bridge process to model the stochastic basis price with the stochastic interest rate and derive the basis price and presale price in a risk-neutral world. Evaluating the partial derivatives of the closed-form expression of the basis and presale prices, we employ sensitivity analysis to investigate the extent to which the initial basis process, the volatility of the basis process, the interest rate, and the time to maturity affect the basis and presale price.

3.1 Stochastic basis process

Comparatively few methods of presale pricing have been used in the literature. Chang and Ward (1993) treat presales as forward contracts and discuss the relationship between the presale price and the spot price. This relationship is the “basis,” defined as the presale price minus the spot price, and can vary based on market organization, variability with regard to quality/location and other product characteristics. Leung et al. (2007) used linear regression to model presale contract price in terms of independent variables such as the market price of existing condominiums, interest rates and hidden forward risks. This paper extends Chang and Ward (1993) and Leung et al. (2007) and proposes an alternative model of presale contract pricing by assuming that the stochastic process of basis to behavior is a modified Brownian bridge process.\(^2\) Furthermore, we consider the changes in the basis price to be affected by the housing price, the interest rate, and the macroeconomic factors at play. Following Yan (2002) and Liu and Longstaff (2004), the basis price \(Z(t,U)\) is defined as the log futures price at

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\(^2\) This study modifies Wang and Wu (2011) and uses a two-factor model to describe the process for futures contracts.
time \( t \) with delivery date \( U \) minus the log spot price at time \( t \). That is,
\[
Z(t, U) = \ln F(t, U) - \ln H(t).
\] (11)

The basis process is assumed to follow a Brownian bridge process, which is used to mimic the basis process directly; we do not indirectly model the basis function using state variables such as tax structure, convenience yield (or dividends), the term structure of the interest rate or stochastic volatility. Under this assumption, the appropriate number of state variables in the futures model and in the basis function does not need to be determined.

A Brownian bridge is a Brownian motion that will reach a specified point at a specified time. Under the assumption of no arbitrage, the housing price and the presale price will converge at the maturity date; thus, the basis, which fluctuates prior to maturity, becomes zero at the maturity date. A Brownian bridge process accounts for the zero value for the basis quite well. However, the volatility of the Brownian bridge process is only a function of time to maturity and cannot reflect the time variation in basis volatility. We follow Wang and Wu (2011) in modifying the Brownian bridge process by time-varying volatility. Let the basis \( Z(t, U) \) be a modified Brownian bridge during the time interval \([0, U]\) with \( Z(0, U) = \ln F(0, U) - \ln H(0) \), and \( Z(0, 0) = 0 \). Under spot martingale measure \( P_Q \), the basis process is defined as follows:

\[
dZ(t, U) = \frac{Z(t, U)}{U - t} dt + \tilde{\sigma}_Z(t, U) \cdot dW_Z^Q(t),
\] (12)

\[
\tilde{\sigma}_Z(t, U) = \sigma_Z(t, U) \times \begin{bmatrix}
\rho_{HZ}
\rho_{MZ}
\rho^*_Z
\end{bmatrix},
\rho^*_Z = \frac{\rho_M - \rho^{**}_{HZ} \rho_{M_H} \rho_{M_H} - \rho_{M_H} \rho_{HM}}{\sqrt{1 - \rho^2_{M_H}} \sqrt{1 - \rho^2_{M_H}}},
\]
\[ \rho_Z^* = \sqrt{1 - \rho_{Hz}^2 - \frac{(\rho_{Hz} - \rho_{Mr} \rho_{tU})^2}{1 - \rho_{Mr}^2} - \rho_{Mz}^2}, \]  

(13)

where \( \sigma_Z(t,U) \) is the time-varying basis volatility, \( \rho_{rZ} (\rho_{Hz}, \rho_{Mz}) \) is the correlation coefficient for the interest rate (the housing price, the macroeconomic factors) and the basis; \( \rho_{Hz} (\rho_{Mr}) \) is the correlation coefficient for the housing price (the macroeconomic factors) and the interest rate; and \( \rho_{IM} \) is the correlation coefficient for the housing price and the macroeconomic factors. \( W^0_Z \) represents one-dimensional Brownian motion defined on the filtered probability space \( (\Omega, F, P_0) \). Based on the specification in equations (12) and (13), the modified Brownian Bridge process considers the basis volatility and allows for interactions among the housing price, the interest rate, the macroeconomic factors and the basis price. Moreover, as shown by Klebaner (1988), \( Z(t,U) \) is equal to zero as \( t \) approaches \( U \). The solution is presented as follows:\(^3\)

\[
Z(t,U) = \frac{(U-t)Z(0,U)}{U} + (U-t) \int_0^t \frac{\dot{\sigma}_Z(t,U) \cdot b(u,U)}{U-u} du + (U-t) \int_0^t \frac{\dot{\sigma}_Z(t,U)}{U-u} dW^T(t).
\]

(14)

According to the above equation, the conditional expected value of the basis process at time \( t \) is

\[
E(\mathcal{F} \cdot t \mid t) = \frac{(U-t)Z(0,U)}{U} + (U-t) \int_0^t \frac{\dot{\sigma}_Z(t,U) \cdot b(u,U)}{U-u} du,
\]

(15)

which is a function of the initial basis \( Z(0,U) \), the volatility of the basis process \( \sigma_Z(t,U) \) and the interest rate \( b(u,U) \) during the time to maturity \( U-t \).

\(^3\) For a similarly detailed derivation technique, see Klebaner (1988, p.121).
3.2 Sensitivity analysis of the basis process

To analyze the characteristics of the basis process, in this subsection, we derive the sensitivities of the conditional expected value of the basis process to the initial basis process, the volatility of the basis process, the interest rate and the time to maturity as follows:

\[
\frac{\partial E(Z(t,U)|F_t)}{\partial Z(0,U)} = \frac{U-t}{U} > 0, \tag{16}
\]

\[
\frac{\partial E(Z(t,U)|F_t)}{\partial \sigma_z(t,U)} = (U-t) \int_0^t \frac{b(u,U)(\rho_{izu} + \rho_{iz} - \rho_{iz} \rho_{iz}^{**} + \rho_{iz}^{**} + \rho_{iz}^{**})}{\sqrt{1 - \rho_{iz}^2}} \frac{du}{U-u}. \tag{17}
\]

\[
\frac{\partial E(Z(t,U)|F_t)}{\partial b(u,U)} = (U-t) \int_0^t \frac{\tilde{\sigma}_z(t,U)}{U-u} du, \tag{18}
\]

\[
\frac{\partial E(Z(t,U)|F_t)}{\partial (U-t)} = \frac{Z(0,U)}{U} + \int_0^t \frac{\tilde{\sigma}_z(t,U) \cdot b(u,U)}{U-u} du - \frac{\tilde{\sigma}_z(t,U) \cdot b(t,U)}{U-t}. \tag{19}
\]

Based on equations (16)–(19), we find that the conditional expected value of the basis process is an increasing function of the initial basis, whereas it is an uncertain function of the volatility of the basis process, the interest rate and the time to maturity. The relationship between the conditional expected value of the basis process and the volatility of the basis process (the interest rate and the time to maturity) depends on the initial basis, the time to maturity and the relationships between the housing price, the interest rate, and the macroeconomic factors. For instance, according to the cost of carry theory, the higher the interest rate and the longer the construction time-lag, the greater the discount required in presales and the greater the basis price (i.e., \( \rho_{iz} > 0 \)) (Chau, Wong, and Yiu, 2003). Furthermore, according to our empirical results as shown in Table 1, \( \rho_{iz} \) is negative. Based on \( \rho_{iz}^{**}, \rho_{iz}^{*}, \rho_{iz} > 0, \rho_{iz} < 0, \) if \( \rho_{iz} > 0, \rho_{iz}^{**} < 0, \)
we have \( \frac{\partial E(Z(t,U)|F_t)}{\partial \sigma_z(t,U)} > 0 \), whereas if \( \rho_{HZ} < 0, \frac{\partial E(Z(t,U)|F_t)}{\partial \sigma_z(t,U)} \) may be negative.

In addition, the conditional variance of the basis at time \( t \) is
\[
\operatorname{Var}(Z(t,U)|F_t) = (U-t)^2 \int_0^t \left( \frac{\tilde{\sigma}_Z(t,U)}{U-u} \right)^2 \, du
\]
depending on the time to maturity and the time-varying volatility function.

### 3.3 Presale pricing model

The closed-form solution obtained using the presale price – basis risk approach is presented in this section. Based on equation (10), the presale \( F(t,U) \) equals \( H(t) \) multiplied by the exponential of \( Z(t,U) \) and satisfies

\[
F(T,U) = f_p(0,T) \exp \left\{ Z(T,U) - \frac{1}{2} \int_0^T |\phi(u,T)|^2 \, du + \int_0^T \phi(u,T) \cdot dW^T(u) \right\}.
\]

The mean and variance of the presale price taking into account the basis risk, which are computed using the moment-generating function method, are as follows:

\[
E(F(T,U)) = \frac{F(0,U)}{B(0,T)} \exp[\gamma(0,T,U)],
\]

\[
\operatorname{Var}(F(T,U)) = \left( \frac{F(0,U)}{B(0,T)} \right)^2 \exp[2\gamma(0,T,U) + \sigma_f^2(0,T,U)] - 1,
\]

where

\[
\gamma(0,T,U) = \frac{-TZ(0,U)}{U} + \frac{1}{2} \sigma_f^2(0,T,U) + \int_0^T \left[ \frac{(U-T)\tilde{\sigma}_Z(t,U) \cdot b(u,U)}{U-u} - \frac{1}{2} |\phi(u,T)|^2 \right] \, du,
\]

\[
\sigma_f^2(0,T,U) = \int_0^T \left| \frac{(U-T)\tilde{\sigma}_Z(t,U)}{U-u} + \phi(u,T) \right|^2 \, du.
\]

Two important characteristics are crucial to the presale price process. First, the expectation
value and the variance in the presale price, respectively are functions of the initial presale price, the initial basis, the housing price volatility, the macroeconomic factors at play and the interest rate. Correlations also exist among the basis, the housing price, the interest rate and the macroeconomic factors. In particular, the variance of the presale price $\sigma^2_F(0,T,U)$ includes two components: (1) the conditional volatility of the basis at time $t$ ($\tilde{\sigma}_Z(t,U)$) depending on the correlations among the basis, housing price, interest rate and macroeconomic factors; and (2) the volatility of the forward housing price return $\phi(u,T)$, which depends on the covariance matrix of the macroeconomic factors, the volatility of the housing price residuals and the relationship between the housing price and the correlation coefficients of the macroeconomic factors. Second, the basis price is both random and zero at maturity.

3.4 Sensitivity analysis of presale price process

To analyze the characteristics of presale pricing given hedging strategies, in this subsection, we derive their sensitivities from the conditional expected value of the presale price relative to the initial presale price, the initial basis process and the variance of the basis process as follows:

$$\frac{\partial E(F(T,U))}{\partial F(0,U)} = \exp\left[\gamma(0,T,U)\right]\frac{\gamma(y(0,T,U))}{B(0,T)} > 0,$$  \hspace{1cm} (23)

$$\frac{\partial E(F(T,U))}{\partial Z(0,U)} = \frac{F(0,U)}{B(0,T)}\exp\left[\gamma(0,T,U)\right]\frac{T}{U} < 0,$$  \hspace{1cm} (24)

$$\frac{\partial E(F(T,U))}{\partial \sigma^2_Z} = \frac{F(0,U)}{B(0,T)}\exp\left[\gamma(0,T,U)\right]\frac{\gamma(y(0,T,U))}{\partial \sigma^2_Z}.$$  \hspace{1cm} (25)

Based on equations (23)~(25), we find that the conditional expected value of the presale
price is an increasing function of the initial presale price, whereas it is a negative function of the initial basis price. In particular, the relationship between the conditional expected value of the presale price and the variance of the basis price depends on the relationships between the housing price, the interest rate and the macroeconomic factors at play.

4. Sensitivity Analysis

4.1 Data description

For the purpose of the sensitivity analysis, the housing price parameters, the macroeconomic factors, the interest rate and the basis should be obtained. For the housing price parameters and macroeconomic factors, we use the estimated parameters and factors in Chang et al. (2012). Chang et al. (2012) use the monthly observations in the S&P/Case-Shiller Home Price Indices as a proxy for housing prices in the U.S. real estate market. They indicate that the significant macroeconomic factors for housing returns are the inflation rate ($INF$), the log changes in land prices ($\Delta \ln LP$), the log changes in construction costs ($\Delta \ln CC$) and the log changes in the stock price index ($\Delta \ln SPI$). The 3-month U.S. Treasury Bill is selected as the interest rate proxy. The interest rates are available from the International Financial Statistics (IFS) database. The sample period, which extends from January 1987 to December 2008, includes 264 monthly observations. We summarize the proxies and estimated values for the housing price, the relevant macroeconomic factors, the interest rate, and the basis in Table 1.

[Table 1 is here].
4.2 The sensitivity analysis for the basis price

The graphical numerical results in Figure 1 and Figure 2 show that the conditional expected value of the basis price is an increasing function of the correlation coefficient for the housing price (macroeconomic factors) and the basis price; however, the sign of the correlation coefficient for the housing price (the macroeconomic factors) and the basis price determines the relationship between the basis volatility and the basis price. When \( \rho_{HZ} \) (\( \rho_{MZ} \)) is -0.5, the conditional expected value of the basis price is negatively related to the basis volatility. When \( \rho_{HZ} \) (\( \rho_{MZ} \)) is positive, the conditional expected value of the basis price is positively related to the basis volatility.

[Figure 1 is here].

[Figure 2 is here].

As indicated by equation (19), the sign of the effect of time to maturity on basis price depends on the ratio of the initial basis to the time to maturity and the relationships between the basis price, the housing price, the interest rate and macroeconomic factors. Figure 3 shows the relationship between the initial basis ratio and the time to maturity, the correlation coefficient for the macroeconomic factors and the basis price, and the effect of time to maturity on the basis price. We find that the effect of time to maturity on the basis price mainly depends on the ratio of the initial basis to the time to maturity. Whether the correlation coefficient for the macroeconomic factors and the basis price is positive or negative, when the ratio of the initial basis to the time to maturity is negative (positive), the effect of the time to maturity on the basis price will be negative (positive).

[Figure 3 is here].
4.3 The sensitivity analysis for presale price

Figures 4 and 5 show that the conditional expected value of the presale price is a decreasing function of the correlation coefficient for the housing price (the macroeconomic factors) and the basis price; however, the sign of the correlation coefficient for the housing price (macroeconomic factors) and the basis price determines the relationship between the basis volatility and the conditional expected value of the presale price. When $\rho_{HZ}$ ($\rho_{MZ}$) is negative, the conditional expected value of the presale price is positively related to the basis volatility. When $\rho_{HZ}$ ($\rho_{MZ}$) is positive, the conditional expected value of the presale price is negatively related to the basis volatility. Figure 6 indicates the relationships between the initial basis, the correlation coefficient for the macroeconomic factors and the conditional expected value of the basis price and the presale price. This figure shows that the conditional expected value of the presale price is a decreasing function of the initial basis. When the presale price is unchanged, a higher initial basis will result in a lower housing price level, which will cause the conditional expected value of the presale price to decrease. Furthermore, the relationship between the conditional expected value of the presale price and the initial basis price is asymmetrical. When a negative initial basis price decreases, the increase in the conditional expected value of the presale price will be greater than when a positive initial basis price increases. Thus, the relative level of the initial basis price has an asymmetrical effect on the conditional expected value of the presale price.

[Figure 4 is here].

[Figure 5 is here].

[Figure 6 is here].
We also compare various models to further examine how the macroeconomic variables affect the variance in the presale price and the conditional expected value of the presale price with basis risk under changes in the initial basis. As shown in equation (21), the variance in the presale price, 

\[ \sigma^2_F(0, T, U) = \int_0^T \left( \frac{(U - T)\tilde{\sigma}_Z(t, U)}{U - u} + \phi(u, T) \right) du , \]

\( \phi(u, T) = \tilde{\sigma}_H + \Omega \sum^r C - b(u, T) \), is influenced by the conditional volatility of the basis price, \( \tilde{\sigma}_Z(t, U) \), and the volatility of the forward housing price return, \( \phi(u, T) \). Hence, we compare the following three cases by various settings of the variance of the presale price:

**Case 1.** We assume that the variance of the presale price includes the conditional volatility of the basis and the volatility of the forward housing price return, i.e.,

\[ \sigma^2_F(0, T, U) = \int_0^T \left( \frac{(U - T)\tilde{\sigma}_Z(t, U)}{U - u} + \phi(u, T) \right) du . \]  

(26)

**Case 2.** We assume that the variance of the presale price includes the conditional volatility of the basis and the volatility of the forward housing price return is considered to be influenced only by its own volatility and not by the effects of the macroeconomic variables on the forward housing price return, i.e., \( \phi(u, T) = \tilde{\sigma}_H - b(u, T) \); we have

\[ \sigma^2_F(0, T, U) = \int_0^T \left( \frac{(U - T)\tilde{\sigma}_Z(t, U)}{U - u} + \tilde{\sigma}_H - b(u, T) \right)^2 du . \]  

(27)

**Case 3.** We assume that the variance of the presale price includes only the volatility of the forward housing price return takes into account only its own volatility, and the conditional volatility of the basis is zero (a constant basis), i.e., \( \tilde{\sigma}_Z(t, U) = 0 \), \( \phi(u, T) = \tilde{\sigma}_H - b(u, T) \); we have
\[ \sigma_r^2(0,T,U) = \int_0^T [\bar{\sigma}_u - b(u,T)]^2 du. \]  

(28)

We show the corresponding conditional expected value of the presale price for each of the three cases in Figure 7. Figure 7 indicates that in each case, when the initial basis price is negative, the conditional expected value of the presale price in Case 2 differs from the value in Case 1; the maximum difference is approximately 1,000 index points. However, when the positive initial basis price increases, the conditional expected value of the presale price in Case 2 is almost the same as in Case 1; the maximum difference is approximately 2 index points. It is possible that the conditional expected value of the presale price in Case 1 is lower than in Case 2 because the negative correlations between the macroeconomic factors (i.e., the rate of change for land prices and the inflation rate, the rate of change for construction costs and the inflation rate, and the stock price index return and the inflation rate) decrease the presale price variance and the conditional expected value of the presale price. Moreover, the differences between the presale prices in Cases 1 and 3 ranges from 27 to 10,000 index points when the initial basis changes from -20 to 60. A lower initial basis yields a greater difference in the conditional expected value of the presale price. Consequently, ignoring the impact of the relevant macroeconomic factors on the basis price would lead us to severely underprice the conditional expected value of the presale price, particularly when the initial basis is lower.

[Figure 7 is here].

5. Conclusions

To the best of our knowledge, no study has determined how presale price changes due to the covariance structure of housing prices and macroeconomic factors and the influence of stochastic basis risk. However, numerous empirical studies show that a significant relationship exists between macroeconomic factors and housing prices. Consequently, this paper extends
Chang and Ward (1993) and Leung et al. (2007) by investigating the relationship between presale price and housing prices using unbiased futures pricing with stochastic basis risk. Using linear regression to model the relationship between housing prices and macroeconomic factors and following Wang and Wu (2011) in assuming that the stochastic basis price follows a modified Brownian bridge process, we provide closed-form solutions for presale price with basis risk. Furthermore, we examine the influence of housing prices, interest rates, and macroeconomic factors on the basis price and presale price.

Based on the empirical results presented by Chang et al. (2012) that indicated that the macroeconomic factors that have a statistically significant impact on housing price returns include the return on land price, construction costs, the inflation rate and the stock price index, our sensitivity findings show that the conditional expected value of the basis price is an increasing function of the correlation coefficient for the housing price (macroeconomic factors) and the basis price. However, the sign of the correlation coefficient determines the relationship between the basis volatility and the conditional expected value of the basis price. Furthermore, the effect of time to maturity on the basis price mainly depends on the ratio of the initial basis to the time to maturity. On the other hand, the conditional expected value of the presale price is a decreasing function of the correlation coefficient for the housing price (macroeconomic factors) and the basis price; however, the sign of that correlation coefficient determines the relationship between the basis volatility and the conditional expected value of the presale price. Finally, the sign of the initial basis and the influence of the relevant macroeconomic factors on basis price also help to determine the presale price. Ignoring the impact of macroeconomic factors (i.e., land price, construction costs, inflation rates and the stock price index) can lead one to underestimate the conditional expected value of the presale price, especially during periods characterized by negative initial basis.
Table 1: Model parameters and their descriptions

<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_H$</td>
<td>Drift</td>
<td>-0.0024</td>
</tr>
<tr>
<td>$\sigma_H$</td>
<td>Standard Deviation of housing price</td>
<td>0.0083</td>
</tr>
<tr>
<td>$\Delta \ln LP$</td>
<td>Land Prices Change Rate</td>
<td>0.7889</td>
</tr>
<tr>
<td>$\Delta \ln CC$</td>
<td>Construction Cost Change Rate</td>
<td>0.6879</td>
</tr>
<tr>
<td>INF</td>
<td>Inflation Rate</td>
<td>0.0057</td>
</tr>
<tr>
<td>$\Delta \ln SPI$</td>
<td>Stock Price Index Return</td>
<td>-0.0123</td>
</tr>
<tr>
<td>$\sigma_{M_1}$</td>
<td>Standard Deviation of Land Prices Change Rate</td>
<td>0.0072</td>
</tr>
<tr>
<td>$\sigma_{M_2}$</td>
<td>Standard Deviation of Construction Cost Change Rate</td>
<td>0.0053</td>
</tr>
<tr>
<td>$\sigma_{M_3}$</td>
<td>Standard Deviation of Inflation Rate</td>
<td>0.3296</td>
</tr>
<tr>
<td>$\sigma_{M_4}$</td>
<td>Standard Deviation of Stock Price Index Return</td>
<td>0.0446</td>
</tr>
<tr>
<td>$\rho_{M_1M_2}$</td>
<td>Correlation of Land Prices Change Rate and Construction Cost Change Rate</td>
<td>0.4045</td>
</tr>
<tr>
<td>$\rho_{M_1M_3}$</td>
<td>Correlation of Land Prices Change Rate and Inflation Rate</td>
<td>-0.3649</td>
</tr>
<tr>
<td>$\rho_{M_1M_4}$</td>
<td>Correlation of Land Prices Change Rate and Stock Price Index Return</td>
<td>0.1754</td>
</tr>
<tr>
<td>$\rho_{M_2M_3}$</td>
<td>Correlation of Construction Cost Change Rate and Inflation Rate</td>
<td>-0.6417</td>
</tr>
<tr>
<td>$\rho_{M_2M_4}$</td>
<td>Correlation of Construction Cost Change Rate and Stock Price Index Return</td>
<td>0.1348</td>
</tr>
<tr>
<td>$\rho_{M_3M_4}$</td>
<td>Correlation of Inflation Rate and Stock Price Index Return</td>
<td>-0.0712</td>
</tr>
</tbody>
</table>

The value of parameters of housing price and the macroeconomic factors follows the estimated parameters in Chang et al. (2012).
Table 1: Model parameters and their descriptions (Cont.)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{r}_r$</td>
<td>the volatility parameter for the interest rate</td>
<td>0.045</td>
</tr>
<tr>
<td>$k$</td>
<td>the mean-reverting force measurement</td>
<td>0.091</td>
</tr>
<tr>
<td>$\theta$</td>
<td>the long-run mean of the interest rate</td>
<td>0.016</td>
</tr>
<tr>
<td>$Z(0,U)$</td>
<td>the initial basis</td>
<td>Variable</td>
</tr>
<tr>
<td>$\sigma_Z$</td>
<td>the volatility of basis</td>
<td>Variable</td>
</tr>
<tr>
<td>$\rho_{rZ}$</td>
<td>the correlation coefficient between the interest rate and the basis</td>
<td>Variable</td>
</tr>
<tr>
<td>$\rho_{HZ}$</td>
<td>the correlation coefficient between the housing price and the basis</td>
<td>Variable</td>
</tr>
<tr>
<td>$\rho_{MZ}$</td>
<td>the correlation coefficient between the macroeconomic factors and the basis</td>
<td>Variable</td>
</tr>
<tr>
<td>$\rho_{Hr}$</td>
<td>the correlation coefficient between the housing price and the interest rate</td>
<td>Variable</td>
</tr>
<tr>
<td>$\rho_{Mr}$</td>
<td>the correlation coefficient between the macroeconomic factors and the interest rate</td>
<td>Variable</td>
</tr>
<tr>
<td>$\rho_{HM}$</td>
<td>the correlation coefficient between the housing price and the macroeconomic factors</td>
<td>Variable</td>
</tr>
<tr>
<td>$F(0,U)$</td>
<td>the initial price of pre-sale</td>
<td>Variable</td>
</tr>
<tr>
<td>$B(0,T)$</td>
<td>the time price of a zero-coupon bond paying one dollar at time $T$</td>
<td>0.97</td>
</tr>
<tr>
<td>$U - T$</td>
<td>the time to maturity of pre-sale</td>
<td>Variable</td>
</tr>
</tbody>
</table>
Figure 1: The impacts of the volatility of basis price and the correlation coefficient between the housing price and the basis price on the conditional expected value of the basis process. The conditional expected value of the basis process in equation (15) are computed by using \( \rho_{z} = 0.2, \rho_{Mz} = 0.2, \rho_{Mr} = -0.2, \rho_{Mr} = 0.2, \rho_{HM} = 0.2, F(0,U) = 115.7, U - T = 25, Z(0,U) = 60 \).

Figure 2: The impacts of the volatility of basis price and the correlation coefficient between the macroeconomic factor and the basis price on the conditional expected value of the basis process. The conditional expected value of the basis process in equation (15) are computed by using \( \rho_{z} = 0.2, \rho_{Mz} = 0.2, \rho_{Mr} = -0.2, \rho_{Mr} = 0.2, \rho_{HM} = 0.2, F(0,U) = 115.7, U - T = 25, Z(0,U) = 60 \).
Figure 3: Relationship among ratio of initial basis to time to maturity, the correlation coefficient between the macroeconomic factor and the basis price and the effect of time to maturity on basis price. The effect of time to maturity on basis price in equation (19) are computed by using \( \rho_{rZ} = -0.2, \rho_{HZ} = -0.2, \rho_{Hr} = -0.2, \rho_{Mr} = -0.2, \rho_{HM} = -0.2, F(0, U) = 115.7 \).

Figure 4: The impacts of the volatility of basis price and the correlation coefficient between the housing price and the basis price on the conditional expected value of the presale price. The conditional expected value of the presale price in equation (21) are computed by using \( \rho_{rZ} = -0.2, \rho_{HZ} = -0.2, \rho_{Hr} = -0.2, \rho_{Mr} = -0.2, \rho_{HM} = -0.2, F(0, U) = 115.7, U - T = 25, Z(0, U) = 60 \).
Figure 5: The impacts of the volatility of basis price and the correlation coefficient between the macroeconomic factor and the basis price on the conditional expected value of the presale price. The conditional expected value of the presale price in equation (21) are computed by using $\rho_Z =-0.2, \rho_{Hz} =-0.2, \rho_{Mr} =-0.2, \rho_{HM} =-0.2, F(0,U) =115.7, U - T =25, Z(0,U) =60$.

Figure 6: The impacts of the initial basis and the correlation coefficient between the macroeconomic factor and the basis price on the conditional expected value of the presale price. The conditional expected value of the presale price in equation (21) are computed by using $\rho_Z =-0.2, \rho_{Hz} =-0.2, \rho_{Mr} =-0.2, \rho_{HM} =-0.2, F(0,U) =115.7, U - T =25$. 
Figure 7: The comparisons of conditional expected value of presale price by different model. The conditional expected value of the presale price in equation (21) are computed by using $\rho_{rZ} = -0.2$, $\rho_{HZ} = -0.2$, $\rho_{HW} = -0.2$, $\rho_{HM} = -0.2$, $F(0,U) = 115.7$, $U - T = 25$. The variance of the presale price in Case 1 is computed by equation (26); the variance of the presale price in Case 2 is computed by equation (27); the variance of the presale price in Case 3 is computed by equation (28).
References

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